



Time Dependent Queuing

Mark S. Daskin

Department of IE/MS, Northwestern University

Evanston, IL 60208

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Outline

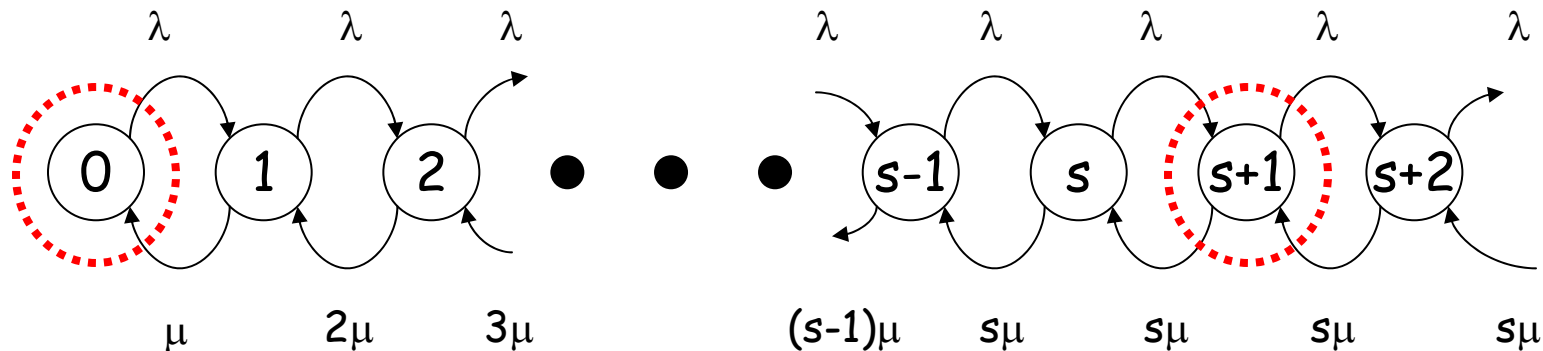
- Will look at $M/M/s$ system
- Numerically integration of Chapman-Kolmogorov equations
- Introduction to Time Dependent Queue Analyzer



Chapman Kolmogorov Equations

– M/M/s queue

State Transition Diagram



$$\frac{d P_0(t)}{dt} = \mu_1(t) P_1(t) - \lambda_0(t) P_0(t)$$

$$\frac{d P_n(t)}{dt} = \hat{\mu}_{n+1}(t) P_{n+1}(t) + \lambda_{n-1}(t) P_{n-1}(t) - [\hat{\mu}_n(t) + \lambda_n(t)] P_n(t) \quad n = 1, \dots$$

$$\hat{\mu}_n(t) = s_n(t) \mu_n(t)$$





First-order difference equations

$$\frac{d P_0(t)}{dt} = \mu_1(t) P_1(t) - \lambda_1(t) P_0(t)$$

$$\frac{P_0(t + \Delta t) - P_0(t)}{\Delta t} = \mu_1(t) P_1(t) - \lambda_1(t) P_0(t)$$

$$P_0(t + \Delta t) = P_0(t) + \Delta t \{ \mu_1(t) P_1(t) - \lambda_1(t) P_0(t) \}$$

$$\frac{d P_n(t)}{dt} = \hat{\mu}_{n+1}(t) P_{n+1}(t) + \lambda_{n-1}(t) P_{n-1}(t) - [\hat{\mu}_n(t) + \lambda_n(t)] P_n(t) \quad n = 1, \dots$$

$$\frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = \hat{\mu}_{n+1}(t) P_{n+1}(t) + \lambda_{n-1}(t) P_{n-1}(t) - [\hat{\mu}_n(t) + \lambda_n(t)] P_n(t) \quad n = 1, \dots$$

$$P_n(t + \Delta t) = P_n(t) + \Delta t \{ \hat{\mu}_{n+1}(t) P_{n+1}(t) + \lambda_{n-1}(t) P_{n-1}(t) - [\hat{\mu}_n(t) + \lambda_n(t)] P_n(t) \} \quad n = 1, \dots$$

You can get first order difference equations as shown on the next slide





First-order difference equations

$$P_0(t + \Delta t) = P_0(t) + \Delta t \{ \mu_1(t) P_1(t) - \lambda_1(t) P_0(t) \}$$

$$P_n(t + \Delta t) = P_n(t) + \Delta t \{ \hat{\mu}_{n+1}(t) P_{n+1}(t) + \lambda_{n-1}(t) P_{n-1}(t) - [\hat{\mu}_n(t) + \lambda_n(t)] P_n(t) \} \quad n = 1, \dots$$

Given some initial estimate of the state probabilities at time t ,

we can use these equations to estimate the state probabilities at some time $t + \Delta t$

and so on....





Practical implementation

- **Make state space finite (max state=N)**
 - Adjust equation for P_N accordingly
- **Divide the day into small time slices**
 - E.g., use $\Delta t=60$ seconds or less.
- **Begin with steady state estimate of probabilities**
 - Increment s as needed to get steady state during any time slice that has $s\mu < \lambda$.





Practical implementation

- Use fourth-order Runge-Kutta to step between time slices and not first-order Euler as shown above
 - If any $P_n(t)$ becomes negative, set it to 0
 - Renormalize all state probabilities at each time period
 - Compute largest % change in probabilities (for $\text{prob} > 0.0001$ for example)





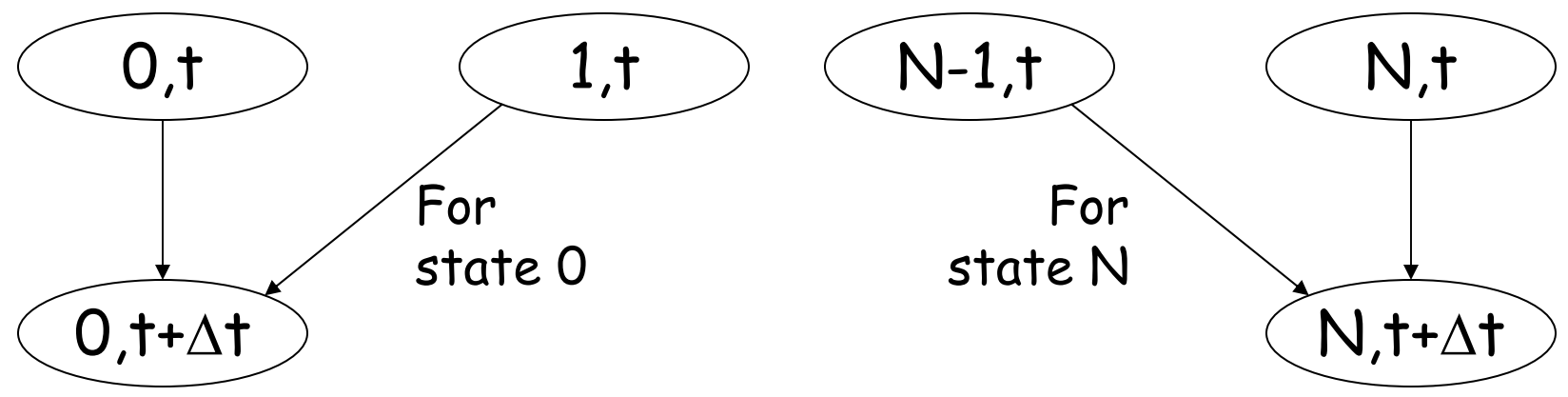
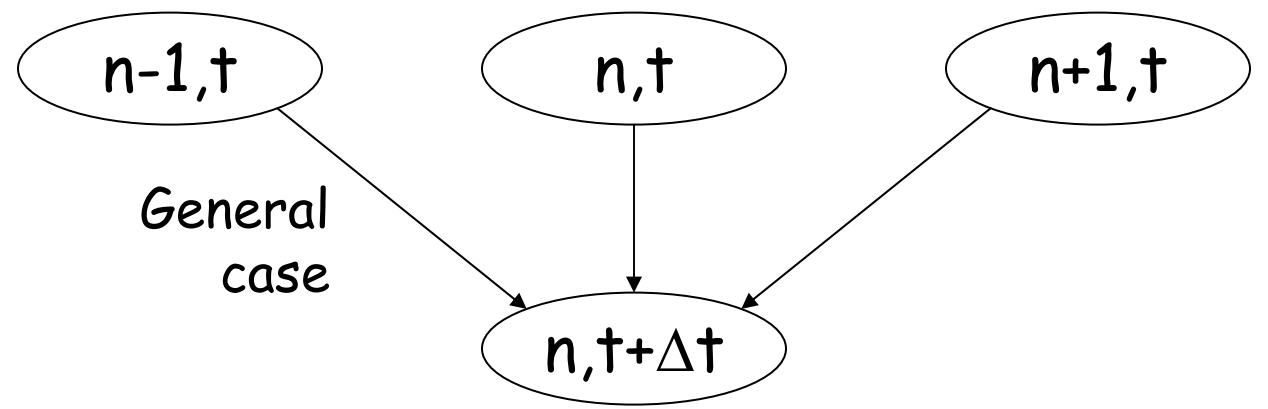
Practical implementation

- If any state probabilities go negative, start process over with smaller Δt
- If largest % change too big, cycle through probabilities again. Repeat as needed.



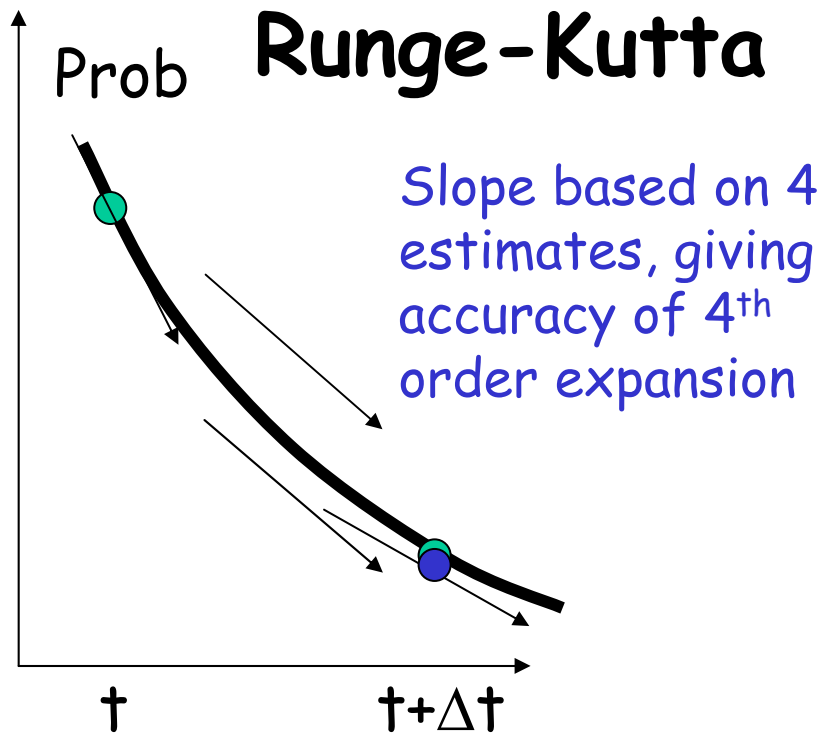
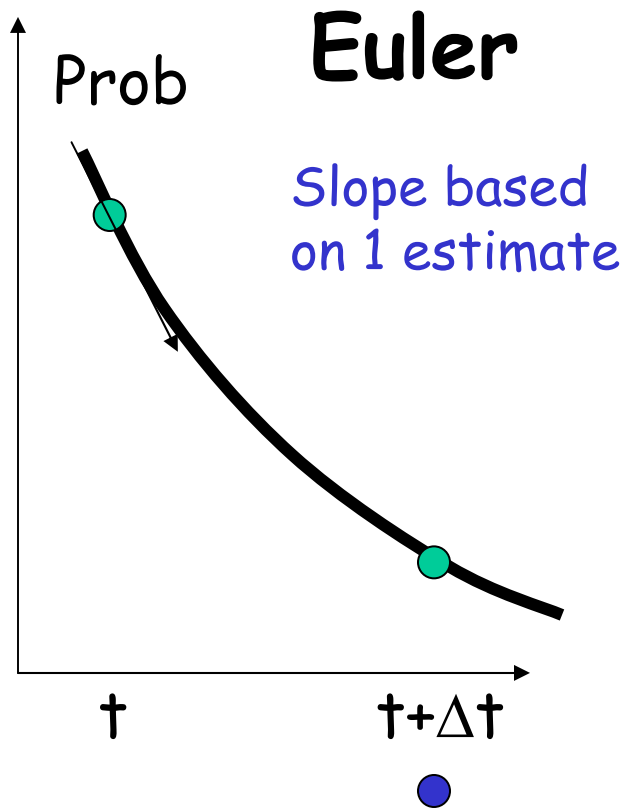


Which state probabilities influence $P_n(t+\Delta t)$





Euler vs 4th Order Runge-Kutta





Euler vs 4th Order Runge-Kutta

Euler $P_n(t + \Delta t) = P_n(t) + f_n[P_n(t), t]\Delta t$

Runge - Kutta $P_n(t + \Delta t) = P_n(t) + [k_1(t) + 2k_2(t) + 2k_3(t) + k_4(t)]\Delta t/3$

$$k_1(t) = f_n[P_n(t), t]\Delta t/2$$

$$k_2(t) = f_n[P_n(t) + k_1(t), t + 0.5\Delta t]\Delta t/2$$

$$k_3(t) = f_n[P_n(t) + k_2(t), t + 0.5\Delta t]\Delta t/2$$

$$k_4(t) = f_n[P_n(t) + 2k_3(t), t + \Delta t]\Delta t/2$$

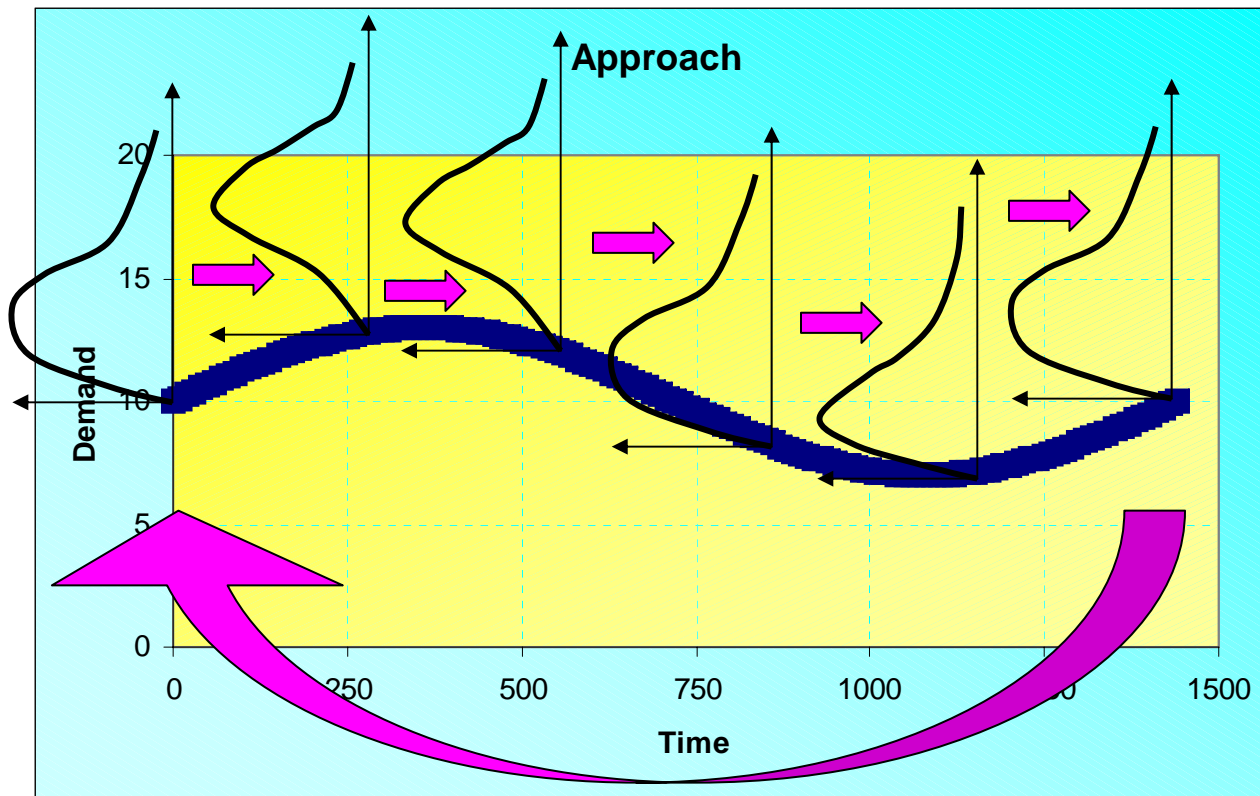
$$f_0(t) = \mu_1(t)P_1(t) - \lambda_1(t)P_0(t)$$

$$f_n(t) = \hat{\mu}_{n+1}(t)P_{n+1}(t) + \lambda_{n-1}(t)P_{n-1}(t) - [\hat{\mu}_n(t) + \lambda_n(t)]P_n(t) \quad n = 1, \dots, N-1$$

$$f_N(t) = \lambda_{N-1}(t)P_{N-1}(t) - \hat{\mu}_N(t)P_N(t)$$



Graphical representation of solution procedure



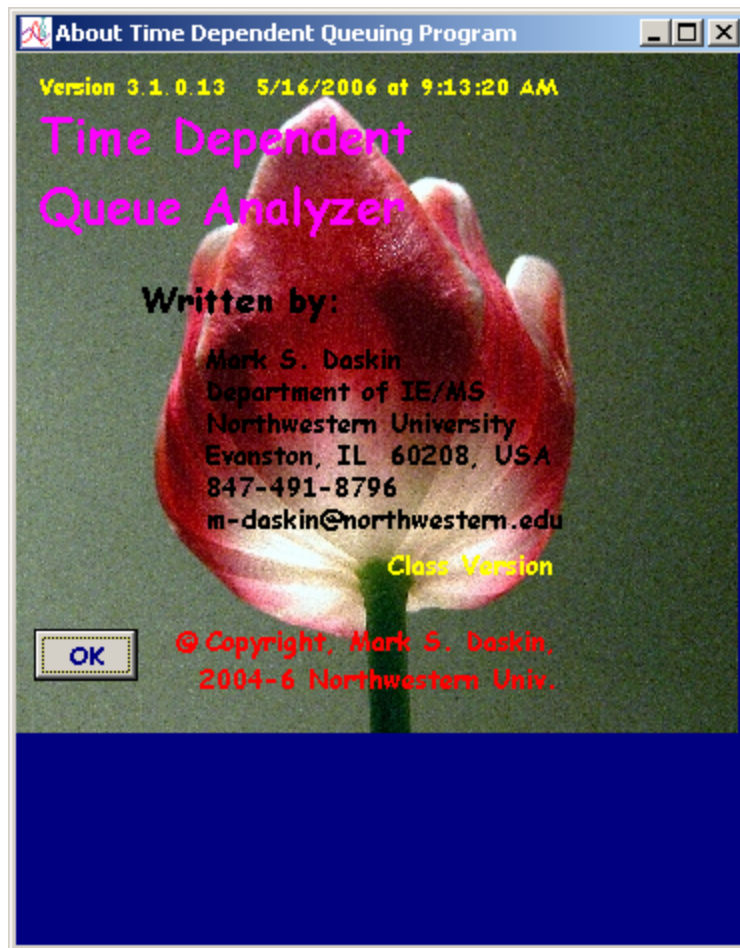
Evolve
PMFs over
time

Cycle
back as
needed





Time Dependent Queue Analyzer



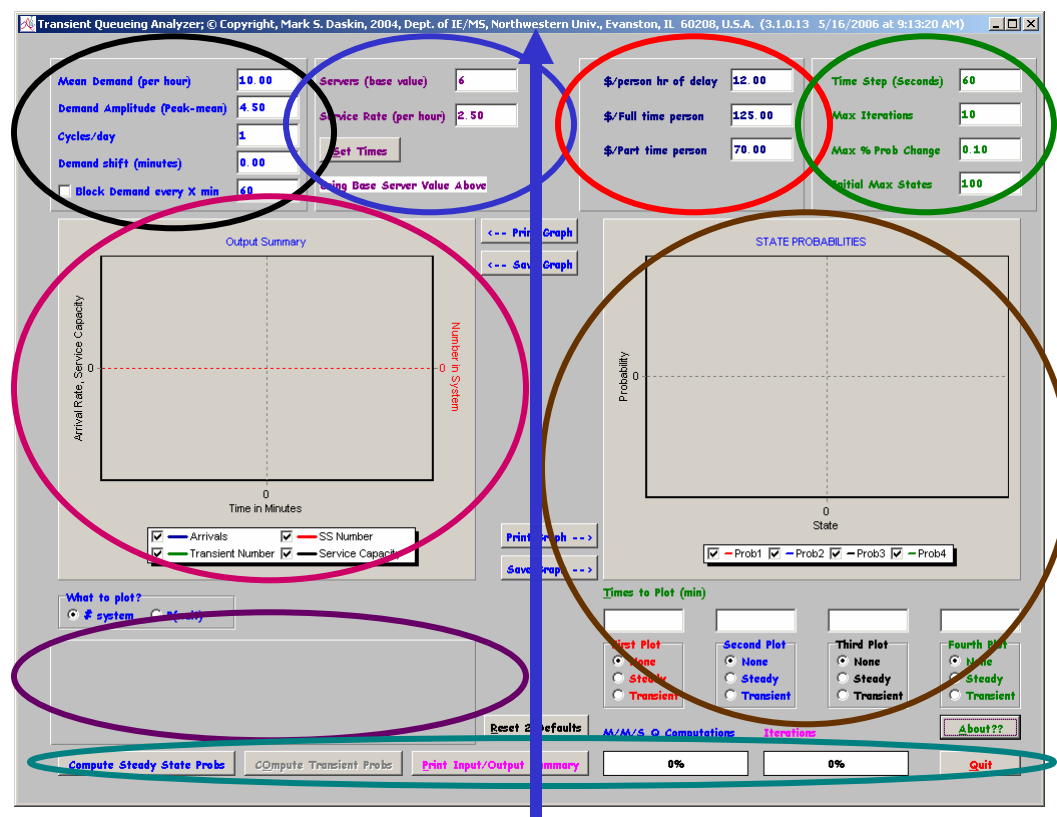
Be sure your copy looks like this.
Earlier versions had an error in the
code.

You should see **Class Version**
underneath my e-mail address.





Major blocks of inputs/outputs



- Demand info
- Service info
- Cost inputs
- Computational info
- Primary graph output
- State prob graphs
- Summary output
- Control Buttons

Be sure your copy has the version number in the top line.
Earlier versions of the code contain an error.





Set times form

Set Number Of Servers

Full Time Duty Schedule | Part Time Duty Schedule | Start Times

GIVEN a FULL TIME employee starts at time 0, indicate the 15 minute intervals (s)he will be on duty

<input checked="" type="checkbox"/> 0 - 15	<input checked="" type="checkbox"/> 240 - 255	<input checked="" type="checkbox"/> 480 - 495	<input type="checkbox"/> 720 - 735	<input type="checkbox"/> 960 - 975	<input type="checkbox"/> 1200 - 1215
<input checked="" type="checkbox"/> 15 - 30	<input checked="" type="checkbox"/> 255 - 270	<input checked="" type="checkbox"/> 495 - 510	<input type="checkbox"/> 735 - 750	<input type="checkbox"/> 975 - 990	<input type="checkbox"/> 1215 - 1230
<input checked="" type="checkbox"/> 30 - 45	<input checked="" type="checkbox"/> 270 - 285	<input checked="" type="checkbox"/> 510 - 525	<input type="checkbox"/> 750 - 765	<input type="checkbox"/> 990 - 1005	<input type="checkbox"/> 1230 - 1245
<input checked="" type="checkbox"/> 45 - 60	<input checked="" type="checkbox"/> 285 - 300	<input checked="" type="checkbox"/> 525 - 540	<input type="checkbox"/> 765 - 780	<input type="checkbox"/> 1005 - 1020	<input type="checkbox"/> 1245 - 1260
<input checked="" type="checkbox"/> 60 - 75	<input checked="" type="checkbox"/> 300 - 315	<input checked="" type="checkbox"/> 540 - 555	<input type="checkbox"/> 780 - 795	<input type="checkbox"/> 1020 - 1035	<input type="checkbox"/> 1260 - 1275
<input checked="" type="checkbox"/> 75 - 90	<input checked="" type="checkbox"/> 315 - 330	<input checked="" type="checkbox"/> 555 - 570	<input type="checkbox"/> 795 - 810	<input type="checkbox"/> 1035 - 1050	<input type="checkbox"/> 1275 - 1290
<input checked="" type="checkbox"/> 90 - 105	<input checked="" type="checkbox"/> 330 - 345	<input checked="" type="checkbox"/> 570 - 585	<input type="checkbox"/> 810 - 825	<input type="checkbox"/> 1050 - 1065	<input type="checkbox"/> 1290 - 1305
<input checked="" type="checkbox"/> 105 - 120	<input checked="" type="checkbox"/> 345 - 360	<input checked="" type="checkbox"/> 585 - 600	<input type="checkbox"/> 825 - 840	<input type="checkbox"/> 1065 - 1080	<input type="checkbox"/> 1305 - 1320
<input checked="" type="checkbox"/> 120 - 135	<input checked="" type="checkbox"/> 360 - 375	<input checked="" type="checkbox"/> 600 - 615	<input type="checkbox"/> 840 - 855	<input type="checkbox"/> 1080 - 1095	<input type="checkbox"/> 1320 - 1335
<input checked="" type="checkbox"/> 135 - 150	<input checked="" type="checkbox"/> 375 - 390	<input checked="" type="checkbox"/> 615 - 630	<input type="checkbox"/> 855 - 870	<input type="checkbox"/> 1095 - 1110	<input type="checkbox"/> 1335 - 1350
<input checked="" type="checkbox"/> 150 - 165	<input checked="" type="checkbox"/> 390 - 405	<input checked="" type="checkbox"/> 630 - 645	<input type="checkbox"/> 870 - 885	<input type="checkbox"/> 1110 - 1125	<input type="checkbox"/> 1350 - 1365
<input checked="" type="checkbox"/> 165 - 180	<input checked="" type="checkbox"/> 405 - 420	<input checked="" type="checkbox"/> 645 - 660	<input type="checkbox"/> 885 - 900	<input type="checkbox"/> 1125 - 1140	<input type="checkbox"/> 1365 - 1380
<input checked="" type="checkbox"/> 180 - 195	<input checked="" type="checkbox"/> 420 - 435	<input checked="" type="checkbox"/> 660 - 675	<input type="checkbox"/> 900 - 915	<input type="checkbox"/> 1140 - 1155	<input type="checkbox"/> 1380 - 1395
<input checked="" type="checkbox"/> 195 - 210	<input checked="" type="checkbox"/> 435 - 450	<input checked="" type="checkbox"/> 675 - 690	<input type="checkbox"/> 915 - 930	<input type="checkbox"/> 1155 - 1170	<input type="checkbox"/> 1395 - 1410
<input checked="" type="checkbox"/> 210 - 225	<input checked="" type="checkbox"/> 450 - 465	<input checked="" type="checkbox"/> 690 - 705	<input type="checkbox"/> 930 - 945	<input type="checkbox"/> 1170 - 1185	<input type="checkbox"/> 1410 - 1425
<input checked="" type="checkbox"/> 225 - 240	<input checked="" type="checkbox"/> 465 - 480	<input checked="" type="checkbox"/> 705 - 720	<input type="checkbox"/> 945 - 960	<input type="checkbox"/> 1185 - 1200	<input type="checkbox"/> 1425 - 1440

Shifts WRAP Around Midnight or 1440 min

Full Time Duty Schedule

Write Starting Times
Read Starting Times
Clear All Starting Times
Exit to Main Window

Duty times

Graph of duty times





Start times page

Time	Arrivals	Min Server	On Duty	FT On	PT On	Full Start	Part Start
0-15	10.27	5	7	7	0		
15-30	10.57	5	7	7	0		
30-45	10.86	5	7	7	0		
45-60	11.15	5	7	7	0		
60-75	11.43	5	7	7	0		
75-90	11.70	5	7	7	0		
90-105	11.97	5	7	7	0		
105-120	12.23	5	7	7	0		
120-135	12.48	5	7	7	0		
135-150	12.72	6	7	7	0		
150-165	12.95	6	7	7	0		
165-180	13.17	6	7	7	0		
180-195	13.37	6	7	7	0		
195-210	13.56	6	7	7	0		
210-225	13.73	6	8	8	0		
225-240	13.89	6	8	8	0		
240-255	14.03	6	8	8	0		
255-270	14.15	6	8	8	0		

Set starting times for full time and part-time employees

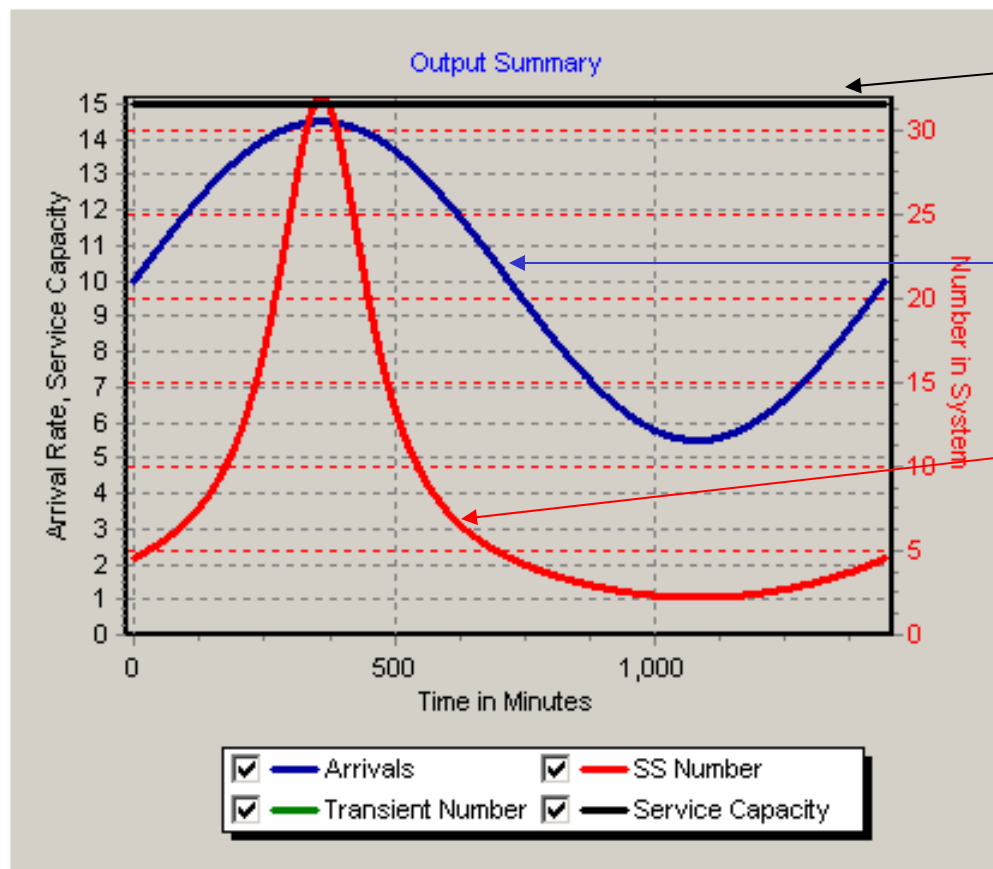
Graph of demand and service capacity

Click to return to main menu





Steady state base case output



Service capacity

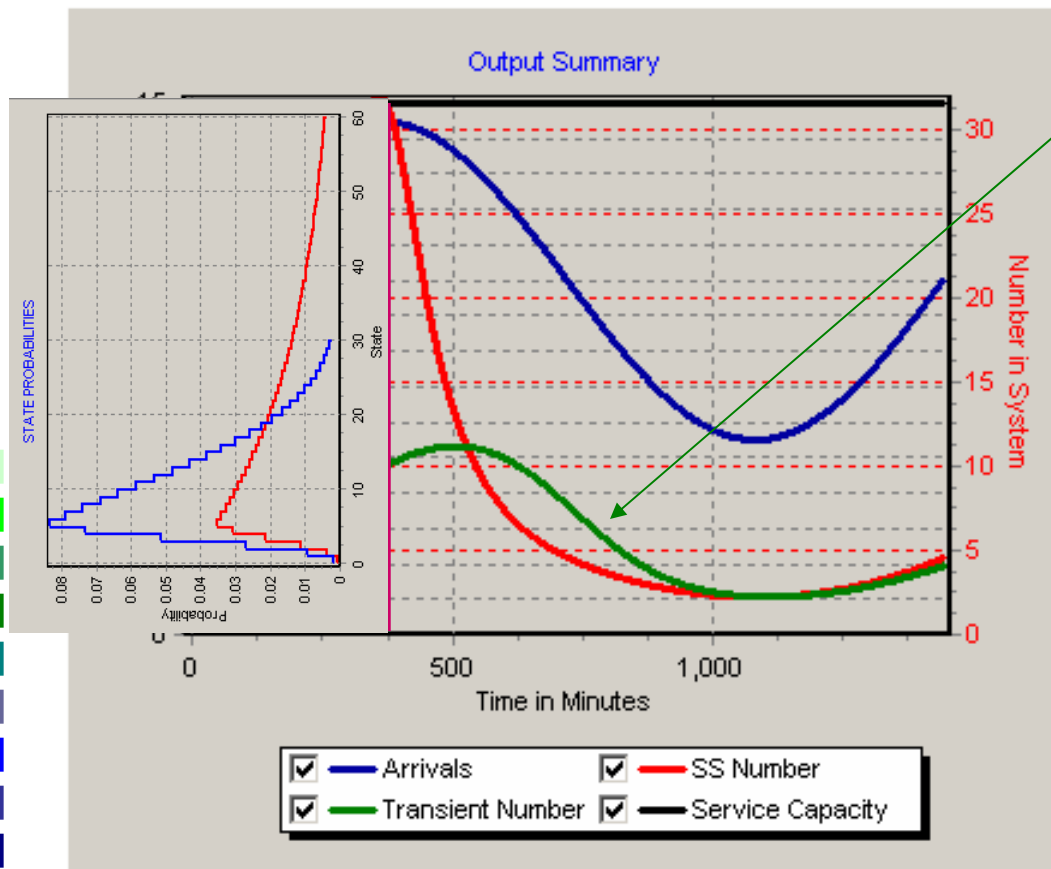
Demand

Mean # in System

Note that peak number in the system corresponds in time to peak demand



Time Dependent Case



Time Dep. # in system

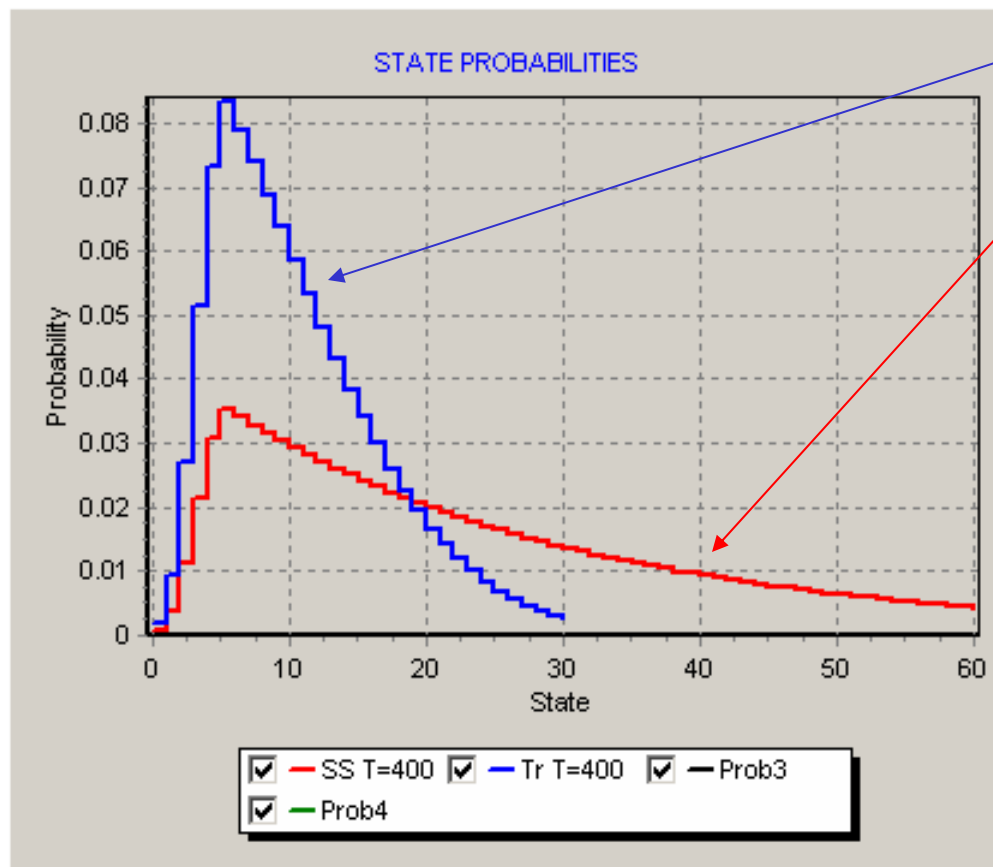
Note that peak number in the system under time dependent conditions is shifted to the right of the peak demand and is lower than then steady-state peak.

Will look at PMF at $t=400$





Comparison of PMFs at $t=400$



Time Dependent

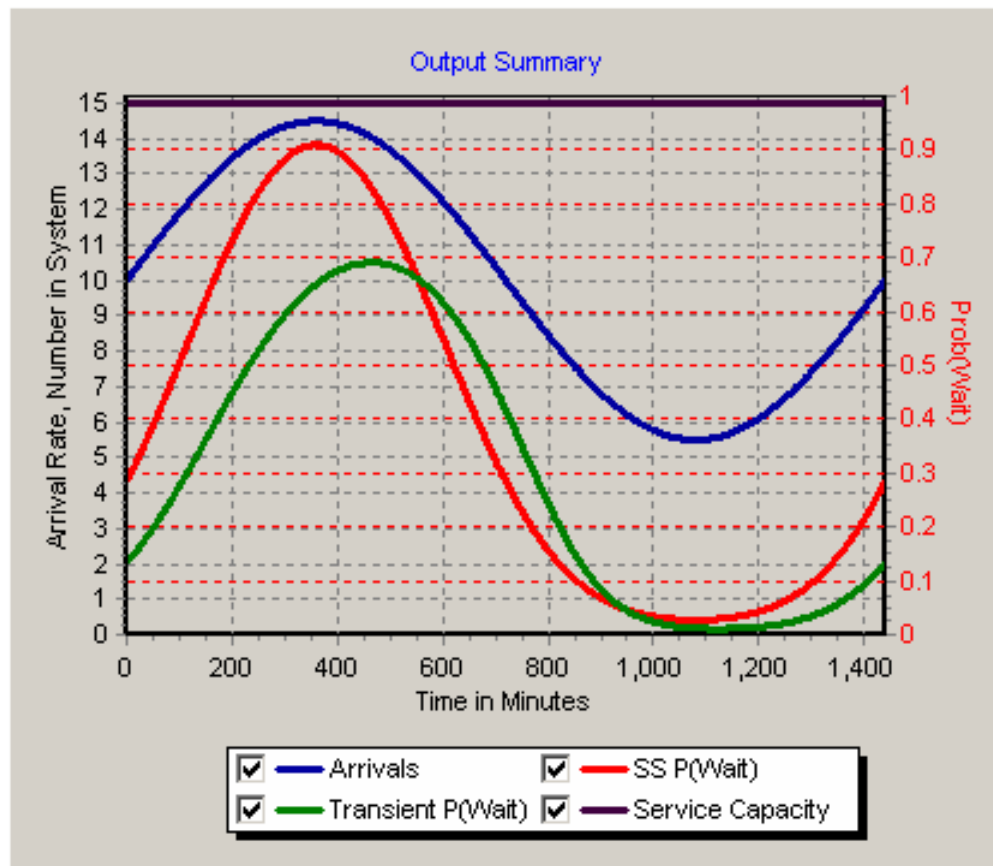
Steady-state

Note that time dependent probabilities are higher at low end reflecting the smaller mean at $t=400$.





P(wait)

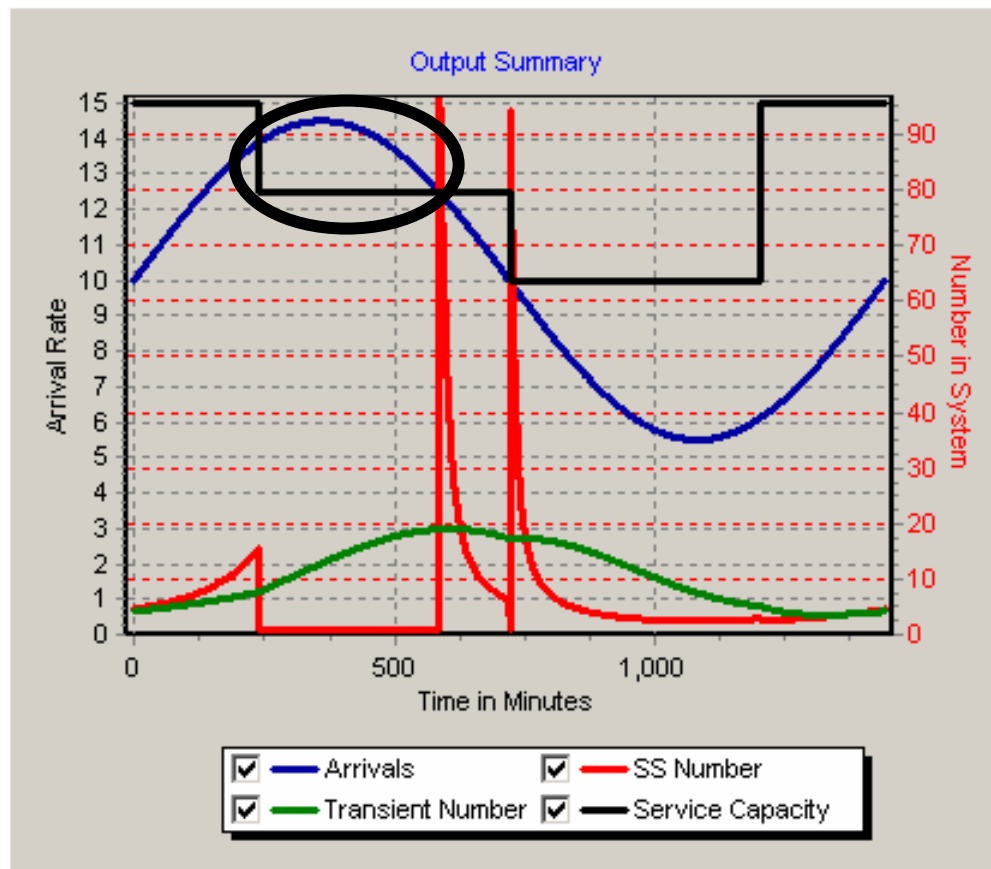


Note that peak P(wait) in the system under time dependent conditions is shifted to the right of the peak demand and is lower than then steady-state peak.





More complex behavior



Note that the program does not require steady state conditions in each time slice.





Summary

- Time dependent queuing analysis is important
- Peaks are shifted to the right of steady state peaks
- Does not require steady-state conditions in all periods.

